

SOLVING SPDES DRIVEN BY COLORED NOISE: A CHAOS APPROACH

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ABSTRACT. An Itô-Skorokhod bi-linear equation driven by infinitely many independent colored noises is considered in a normal triple of Hilbert spaces. The special feature of the equation is the appearance of the Wick product in the definition of the Itô-Skorokhod integral, requiring innovative approaches to computing the solution. A chaos expansion of the solution is derived and several truncations of this expansion are studied. A recursive approximation of the solution is suggested and the corresponding approximation error bound is computed.

1. INTRODUCTION

Stochastic differential equations driven by Gaussian white noise are well-studied; see, for example, the book [35] for ordinary differential equation and the book [37] for equations with partial derivatives. The underlying stochastic process in these equations is the standard Brownian motion W , which is a square-integrable Gaussian martingale with continuous trajectories and independent increments. A lot less is known about equations driven by *colored noise*, when the underlying process is still Gaussian, but no longer has independent increments. An important example is the fractional Brownian motion W^H , $H \in (0, 1)$, which coincides with the standard Brownian motion W for $H = 1/2$ and is not a semi-martingale for all $H \neq 1/2$. It is the lack of the semi-martingale property that makes the analysis difficult at the very basic level, the definition of the corresponding stochastic integral. Several versions of the stochastic integral with respect to W^H have been proposed [1, 12, 13, 14, 23, 26, 38]. Unlike the standard Brownian motion, different approaches such as Itô-type vs. Stratonovich-type integral or path-wise vs. mean-square definition, become much more difficult to reconcile. The paper by V. Pipiras and M. Taqqu [36] describes the main difference between integration with respect to W and W^H for $H \neq 1/2$, and demonstrates the resulting technical difficulty. Beside a purely theoretical interest, fractional Brownian motion appears to be a natural replacement of the standard Brownian motion in certain applied problems [3, 4, 5, 9, 19, 22]. In many such problems, it is possible to avoid most of the technical issues related to the definition of the stochastic integral by considering additive noise.

2000 *Mathematics Subject Classification.* Primary 60H15; Secondary 35R60, 60H40.

Key words and phrases. Generalized Random Fields, Malliavin Calculus, Skorokhod Integral, Wiener Chaos.

S. V. Lototsky acknowledges support from the NSF CAREER award DMS-0237724.

The work of K. Stemann was partially supported by the NSF Grant DMS-0237724.

One way to streamline the analysis of differential equations driven by multiplicative noise (also known as **bilinear equations**) is to use the Wick product. It is known [18] that, with standard Brownian motion, Wick product and usual calculus lead to the same results as the usual product and the Itô calculus. While the use of Wick product has been questioned as a modeling tool for certain applications in economics and finance [8], it is still an effective tool for theoretical investigations, corresponding to the Itô-Skorokhod integral in the white noise analysis.

Successful study of differential equations driven by multiplicative colored noise also requires a convenient representation of the underlying Gaussian process. Traditionally, a Gaussian process is defined by its mean and covariance functions, but then the definition of the integral immediately leads to a number of technical conditions on these functions [2]. An alternative definition is possible [29] by combining the ideas from the theory of generalized Gaussian fields [15, 30], the white noise theory [17, 18], and the Malliavin calculus [31, 34]. This approach to stochastic integration is used in this paper and is outlined in Section 2

Numerical methods for stochastic ordinary differential equations driven by white noise are a well-developed subject [24, 32]. Equations with partial derivatives have been mostly studied in connection with optimal nonlinear filtering [6, 7, 16, 20, 21]. In the end, all these numerical methods have obvious counterparts in the numerical analysis of the deterministic equations (Galerkin method, Euler method, finite difference and finite element methods, operator splitting method, etc.)

The main difficulty in the numerical analysis of equations driven by colored noise is the use of the Wick product: unlike the usual product, Wick product is not an operation readily performed by a computer. Since the Wick product is relatively easy to compute for Hermite polynomials of Gaussian random variables, an implementation of this operation should be based on the chaos expansion, and then a truncation of this expansion becomes a natural numerical approximation of the solution.

While a truncated chaos expansion has been investigated before as an approximation of the solution of a stochastic equation, in particular, for equations of optimal nonlinear filtering [10, 27, etc.], this approximation was always just another possibility of solving the equation numerically. By contrast, for equations driven by colored noise, chaos expansion appears to be the only possibility to compute the solution. The chaos expansion of the solution of a bi-linear equation driven by infinitely many independent colored noises is studied below in Section 3; various truncations of this expansion are studied in Section 4.

2. COLORED NOISE

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

Definition 2.1. (a) A **colored noise** \mathfrak{X} on $L_2((0, T))$ with the covariance operator \mathcal{R} is a collection of zero-mean Gaussian random variables $\mathfrak{X}(f)$, $f \in L_2((0, T))$, with

the property

$$(2.1) \quad \mathbb{E}(\mathfrak{X}(f)\mathfrak{X}(g)) = \int_0^T (\mathcal{R}f)(s)g(s)ds, \quad f, g \in L_2((0, T)),$$

where \mathcal{R} is a bounded linear operator on $L_2((0, T))$.

(b) A **representation operator** of \mathfrak{X} is a bounded linear operator \mathcal{K} on $L_2((0, T))$ such that $\mathcal{K}\mathcal{K}^* = \mathcal{R}$, where \mathcal{K}^* is the adjoint of \mathcal{K} .

Remark 2.2. (a) The white noise on $L_2((0, T))$ corresponds to $\mathcal{R} = I$, the identity operator [17].

(b) Since \mathcal{R} is a self-adjoint non-negative operator, a representation operator always exists.

For $t > 0$ let

$$\chi_t(s) = \begin{cases} 1, & 0 \leq s \leq t, \\ 0, & \text{otherwise,} \end{cases}$$

denote the characteristic function of the interval $[0, t]$.

Example 2.3. If \mathfrak{X} is white noise, then direct computations show that $W(t) = \mathfrak{X}(\chi_t)$ is a standard Brownian motion and

$$(2.2) \quad \mathfrak{X}(f) = \int_0^T f(s)dW(s), \quad f \in L_2((0, T)).$$

Example 2.4. (Ornstein-Uhlenbeck Noise)

For $b > 0$ and $t, s \in [0, T]$ define

$$K(t, s) = -be^{-b(t-s)}.$$

Then the operator

$$\mathcal{K} : f(t) \mapsto (\mathcal{K}f)(t) = f(t) + \int_0^t K(t, s)f(s)ds$$

is bounded on $L_2((0, T))$:

$$\int_0^T |(\mathcal{K}f)(s)|^2 ds \leq (1 + \sqrt{bT})^2 \int_0^T |f(s)|^2 ds.$$

If \mathfrak{X} is the colored noise corresponding to the representation operator \mathcal{K} , then a straightforward computation shows that $\mathfrak{X}(\chi_t)$, $t \in (0, T)$, is the Ornstein-Uhlenbeck process with covariance

$$\mathbb{E}(\mathfrak{X}(\chi_t)\mathfrak{X}(\chi_s)) = e^{-b|t-s|}.$$

Accordingly, we call \mathfrak{X} the **Ornstein-Uhlenbeck noise** with parameter b .

Example 2.5. (Fractional White Noise)

For $H \in (1/2, 1)$ and $t, s \in [0, T]$ define

$$K(t, s) = C_H \left(H - \frac{1}{2} \right) \left(\frac{t}{s} \right)^{\frac{1}{2}-H} (t-s)^{H-\frac{3}{2}} \chi_t(s),$$

where

$$C_H = \left(\frac{2H\Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2})\Gamma(2 - 2H)} \right)^{\frac{1}{2}}$$

and Γ is the Gamma function. Then the operator

$$\mathcal{K} : f(t) \mapsto (\mathcal{K}f)(t) = \int_0^t K(t, s)f(s)ds$$

is bounded on $L_2((0, T))$ [29, 33]:

$$(2.3) \quad \int_0^T |(\mathcal{K}f)(s)|^2 ds \leq \frac{H(2H - 1)\Gamma(H - \frac{1}{2})}{\Gamma(H + \frac{1}{2})} T^{2H-1} \int_0^T |f(s)|^2 ds.$$

If \mathfrak{X} is the colored noise corresponding to the representation operator \mathcal{K} , then $\mathfrak{X}(\chi_t)$, $t \in (0, T)$, is the fractional Brownian motion with Hurst parameter H [33]. Accordingly, we call \mathfrak{X} the H -fractional white noise.

In general, if \mathfrak{X} is a colored noise on $L_2((0, T))$, then $X(t) = \mathfrak{X}(\chi_t)$ is a zero-mean Gaussian process. Thus, one can interpret \mathfrak{X} as a collection of integrals $\int_0^T f(s)dX(s)$ for *deterministic* $f \in L_2((0, T))$. Our study of bi-linear equations (linear equations with multiplicative noise) requires an extension of \mathfrak{X} to *random* f and is based on the following generalization of (2.2).

Proposition 2.6. For every colored noise \mathfrak{X} on $L_2((0, T))$ with a representation operator \mathcal{K} , there exists a unique standard Brownian motion $W = W(t)$ such that

$$(2.4) \quad \mathfrak{X}(f) = \int_0^T (\mathcal{K}^* f)(s)dW(s), \quad f \in L_2((0, T)).$$

Proof. Relation (2.4) certainly defines a colored noise on $L_2((0, T))$; we omit the technical proof that the corresponding Brownian motion can be found for every colored noise \mathfrak{X} [29]. \square

Definition 2.7. A pair (\mathcal{K}, W) , where \mathcal{K} is a bounded linear operator on $L_2((0, T))$ and W is a standard Brownian motion, is called a **representation** of the colored noise \mathfrak{X} if (2.4) holds.

For random f , we now define $\mathfrak{X}(f)$ according to (2.4), where the stochastic integral is understood in the **Itô-Skorokhod sense** [34]. A equivalent, but less convenient, definition of $\mathfrak{X}(f)$ for random f is possible in intrinsic terms without using a representation of \mathfrak{X} [29].

3. CHAOS SOLUTION: EXISTENCE AND REGULARITY

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and \mathfrak{X}_ℓ , $\ell \geq 1$, a collection of independent colored noises on $L_2((0, T))$.

In this section we study the equation

$$(3.1) \quad u(t) = u_0 + \int_0^t (\mathcal{A}u(s) + F(s))ds + \sum_{\ell=1}^{\infty} \mathfrak{X}_\ell(\chi_t \mathcal{B}_\ell u + \chi_t G_\ell),$$

$t \in [0, T]$, in a normal triple $(\mathbf{X}, \mathbf{H}, \mathbf{X}')$ of Hilbert spaces. In particular, we assume that \mathcal{A} and each \mathcal{B}_ℓ are bounded linear operators from \mathbf{X} to \mathbf{X}' , $u_0 \in L_2(\Omega; \mathbf{X}')$, $F, G_\ell \in L_2(\Omega; L_2((0, T); \mathbf{X}'))$.

Remark 3.1. While it is tempting to re-write (3.1) as

$$(3.2) \quad u(t) = u_0 + \int_0^t (\mathcal{A}u(s) + F(s))ds + \sum_{\ell=1}^{\infty} \int_0^t (\mathcal{B}_\ell u(s) + G_\ell(s))dX_\ell(s),$$

where $X_\ell(t) = \mathfrak{X}_\ell(\chi_t)$, we will use a more rigorous form (3.1).

By analogy with equations driven by white noise [37], we define a **variational solution** of (3.1) as a random element u with values in $L_2(\Omega; L_2((0, T); \mathbf{X}))$ such that, for every $v \in \mathbf{X}$, the equality

$$(3.3) \quad (u(t), v)_{\mathbf{H}} = \langle u_0, v \rangle + \int_0^t \langle (\mathcal{A}u(s) + F(s)), v \rangle ds + \sum_{\ell=1}^{\infty} \mathfrak{X}_\ell(\chi_t \langle \mathcal{B}_\ell u + G_\ell, v \rangle)$$

holds in \mathbf{X}' on the same set of probability one for all $t \in [0, T]$; by $\langle \cdot, \cdot \rangle$ we denote the duality between \mathbf{X}' and \mathbf{X} relative to the inner product $(\cdot, \cdot)_{\mathbf{H}}$ of \mathbf{H} .

Unfortunately, the current development of the colored noise calculus is not sufficient to establish existence of a variational solution of (3.1). Accordingly, we introduce a weaker notion of solution, called a *chaos solution*, using a Fourier series expansion in the space of square integrable random processes.

We start with some auxiliary constructions. Let $(\mathcal{K}_\ell, W_\ell)$ be a representation of the colored noise \mathfrak{X}_ℓ and let $\{h_k, k \geq 1\}$ be an orthonormal basis in $L_2((0, T))$. Define random variables

$$(3.4) \quad \xi_{k\ell} = \int_0^T h_k(t) dW_\ell(t).$$

Let \mathcal{J} be the collection of multi-indices $\alpha = \{\alpha_{k\ell}, k, \ell \geq 1\}$. Each $\alpha \in \mathcal{J}$ has non-negative integer elements $\alpha_{k\ell}$ and $\sum_{k,\ell} \alpha_{k\ell} < \infty$. We will use the notations

$$(3.5) \quad |\alpha| = \sum_{k,\ell} \alpha_{k\ell}, \quad \alpha! = \prod_{k,\ell} \alpha_{k\ell}!.$$

By (0) we denote the multi-index α with $|\alpha| = 0$ and by ϵ_{ij} , the multi-index α with $|\alpha| = 1$ and $\alpha_{ij} = 1$.

Remark 3.2. If there is only one colored noise, then the entries of α have only one index: $\alpha = \{\alpha_k, k \geq 1\}$.

For $\alpha \in \mathcal{J}$ define

$$(3.6) \quad \xi_\alpha = \prod_{k,\ell \geq 1} \frac{H_{\alpha_{k\ell}}(\xi_{k\ell})}{\sqrt{\alpha_{k\ell}!}},$$

where, for an integer $n \geq 0$, $H_n = H_n(t)$ is the n -th Hermite polynomial

$$(3.7) \quad H_n(t) = (-1)^n e^{t^2/2} \frac{d^n}{dt^n} e^{-t^2/2}.$$

Recall the definition of the Wick product [18]:

$$(3.8) \quad H_m(\xi_{ij}) \diamond H_n(\xi_{k\ell}) = \begin{cases} H_{m+n}(\xi_{ij}), & \text{if } i = k \text{ and } j = \ell \\ H_m(\xi_{ij}) H_n(\xi_{k\ell}), & \text{otherwise.} \end{cases}$$

In particular, for $m \geq 1$,

$$(3.9) \quad H_m(\xi_{k\ell}) = \underbrace{\xi_{k\ell} \diamond \cdots \diamond \xi_{k\ell}}_{m \text{ times}}.$$

Then every multi-index α is uniquely characterized by the collection $\{(k_1, \ell_1), \dots, (k_n, \ell_n)\}$, called the **characteristic set** of α , such that $k_1 \leq k_2 \leq \cdots \leq k_n$, $\ell_i \leq \ell_{i+1}$ if $k_i = k_{i+1}$, and

$$(3.10) \quad \xi_\alpha = \frac{\xi_{k_1 \ell_1} \diamond \xi_{k_2 \ell_2} \diamond \cdots \diamond \xi_{k_n \ell_n}}{\sqrt{\alpha!}}.$$

Proposition 3.3. (a) The collection $\{\xi_\alpha, \alpha \in \mathcal{J}\}$ is an orthonormal basis in the space of square integrable random variables that are measurable with respect to the σ -algebra \mathcal{F}^W generated by the Brownian motions W_ℓ , $\ell \geq 1$, on $[0, T]$.

(b) Let η be a square-integrable \mathcal{F}^W -measurable random element with values in $L_2((0, T))$. Define

$$(3.11) \quad \eta_\alpha(t) = \mathbb{E}(\eta_k(t) \xi_\alpha).$$

Then

$$(3.12) \quad \mathfrak{X}_\ell(\eta) = \sum_{\alpha \in \mathcal{J}} \left(\sum_{k \geq 1} \sqrt{\alpha_{k\ell}} \int_0^T \eta_{\alpha - \epsilon_{k\ell}}(s) (\mathcal{K}_\ell m_k)(s) ds \right) \xi_\alpha$$

as long as the series in α converges in the mean square (the inner sum always contains finitely many non-zero terms).

Proof. Part (a) is a classical result of Cameron and Martin [11]. Part (b) follows from the definition of the Itô-Skorokhod integral in terms of the Wick product [29]. □

Let us now assume that equation (3.1) has a variational solution u and

$$(3.13) \quad u(t) = \sum_{\alpha \in \mathcal{J}} u_\alpha(t) \xi_\alpha.$$

Substituting this representation into (3.1) and using (3.12), we conclude that each u_α , which is non-random, satisfies

$$(3.14) \quad \begin{aligned} u_\alpha(t) = & u_{0,\alpha} + \int_0^t (\mathcal{A}u_\alpha(s) + F_\alpha(s))ds \\ & + \sum_{k,\ell=1}^{\infty} \sqrt{\alpha_{k\ell}} \int_0^t (\mathcal{B}_\ell u_{\alpha-\epsilon_{k\ell}}(s) + G_{\ell,\alpha-\epsilon_{k\ell}}(s))(\mathcal{K}_\ell h_k)(s)ds, \end{aligned}$$

where $u_{0,\alpha} = \mathbb{E}(u_0 \xi_\alpha)$, $F_\alpha(t) = \mathbb{E}(F(t) \xi_\alpha)$, $G_{\ell,\alpha}(t) = \mathbb{E}(G_\ell(t) \xi_\alpha)$. This observation motivates the following definition of the chaos solution.

Definition 3.4. (a) The collection of functions $\{u_\alpha, \alpha \in \mathcal{J}\}$ is called a **chaos solution** of equation (3.1) if every u_α is an element of $L_2((0, T); \mathbf{X})$ and the system of equalities (3.14) holds in \mathbf{X}' for all $t \in [0, T]$. The chaos solution is called **square integrable** if

$$(3.15) \quad \sup_{0 < t < T} \sum_{\alpha \in \mathcal{J}} \|u_\alpha(t)\|_{\mathbf{H}}^2 < \infty.$$

(b) The system of equalities (3.14) is called the **S-system** corresponding to equation (3.1).

Remark 3.5. (a) If $\{u_\alpha, \alpha \in \mathcal{J}\}$ is a square integrable chaos solution, then, for each $t \in [0, T]$, $u(t) = \sum_{\alpha \in \mathcal{J}} u_\alpha(t) \xi_\alpha$ is an element of $L_2(\Omega; \mathbf{H})$, but still there is no guarantee that u is a variational solution.

(b) Uniqueness of the chaos solution implies uniqueness of the variational solution.

To establish existence and uniqueness of the chaos solution, we look at (3.1) as a system of equations. To solve this system, we make the following assumptions:

A1 The operator \mathcal{A} is bounded linear from \mathbf{X} to \mathbf{X}' and is **strongly parabolic**: there exist a positive number δ_A and a real number C_A such that, for all $v \in \mathbf{X}$,

$$(3.16) \quad \langle \mathcal{A}v, v \rangle + \delta_A \|v\|_{\mathbf{X}}^2 \leq C_A \|v\|_{\mathbf{H}}^2.$$

A2 Each \mathfrak{X}_ℓ , $\ell \geq 1$, is a colored noise on $L_2((0, T))$ with a representation operator \mathcal{K}_ℓ and each \mathcal{K}_ℓ is a bounded linear operator on $L_2((0, T))$ with the operator norm \mathfrak{K}_ℓ :

$$(3.17) \quad \int_0^T |(\mathcal{K}_\ell f)(s)|^2 ds \leq \mathfrak{K}_\ell^2 \int_0^T f^2(s) ds, \quad f \in L_2((0, T)).$$

A3 The functions u_0, F, G_ℓ , $\ell \geq 1$, are non-random and \mathfrak{X}_ℓ , $\ell \geq 1$, are jointly independent.

A4 Each \mathcal{B}_ℓ is a bounded linear operator on \mathbf{H} with the operator norm C_ℓ :

$$(3.18) \quad \|\mathcal{B}_\ell v\|_{\mathbf{H}} \leq C_\ell \|v\|_{\mathbf{H}}, \quad v \in \mathbf{H}.$$

A5 The following regularity conditions hold:

$$(3.19) \quad I_0 = \|u_0\|_{\mathbf{H}}^2 + \frac{2}{\delta_A} \int_0^T \|F(t)\|_{\mathbf{X}}^2 dt + \sum_{\ell=1}^{\infty} \mathfrak{K}_\ell^2 \int_0^T \|G_\ell(t)\|_{\mathbf{H}}^2 dt < \infty,$$

$$(3.20) \quad C_B = \sum_{\ell=1}^{\infty} \mathfrak{K}_{\ell}^2 C_{\ell}^2 < \infty.$$

Theorem 3.6. *Under Assumptions **A1**–**A5** equation (3.1) has a unique chaos solution. The solution is square integrable and satisfies*

$$(3.21) \quad \sup_{0 < t < T} \mathbb{E} \|u(t)\|_{\mathbf{H}}^2 \leq C_o e^{(C_A + C_B)T} I_0,$$

where $C_A > 0$ is from (3.16) and $1 \leq C_o \leq 3$. In particular, $C_o = 1$ if $F = G_{\ell} = 0$.

Proof. Under Assumption **A3** the the S-system (3.14) corresponding to equation (3.1) becomes

$$(3.22) \quad \begin{aligned} u_{(0)}(t) &= u_0 + \int_0^t \mathcal{A}u_{(0)}(s)ds + \int_0^t F(s)ds, \quad |\alpha| = 0; \\ u_{\epsilon_{ij}}(t) &= \int_0^t \mathcal{A}u_{\epsilon_{ij}}(s)ds + \int_0^t \left(\mathcal{B}_j u_{(0)}(s) + G_j(s) \right) (\mathcal{K}_j h_i)(s)ds, \quad |\alpha| = 1; \\ u_{\alpha}(t) &= \int_0^t \mathcal{A}u_{\alpha}(s)ds + \sum_{k, \ell=1}^{\infty} \sqrt{\alpha_{k\ell}} \int_0^t \mathcal{B}_{\ell} u_{\alpha - \epsilon_{k\ell}}(s) (\mathcal{K}_{\ell} h_k)(s)ds, \quad |\alpha| > 1. \end{aligned}$$

Indeed, by Assumption **A3**, if $|\alpha| > 0$, then $\mathbb{E}(u_0 \xi_{\alpha}) = 0$, $\mathbb{E}(F(t) \xi_{\alpha}) = 0$, and $\mathbb{E}(G_{\ell} \xi_{\alpha}) = 0$.

The following proposition provides a key estimate for the solution of this system and is the main step in the proof of Theorem 3.6.

Proposition 3.7. Under Assumptions **A1**–**A6**, for every $0 \leq t \leq T$ and $k \geq 1$,

$$(3.23) \quad \begin{aligned} \sum_{\substack{\alpha \in \mathcal{J} \\ |\alpha|=k}} \|u_{\alpha}(t)\|_{\mathbf{H}}^2 &\leq C_o e^{C_A T} \left(\frac{(C_B T)^k}{k!} \left(\|u_0\|_{\mathbf{H}}^2 + \frac{2}{\delta_A} \int_0^T \|F(s)\|_{\mathbf{X}'}^2 ds \right) \right. \\ &\quad \left. + \frac{(C_B T)^{k-1}}{(k-1)!} \sum_{\ell=1}^{\infty} \mathfrak{K}_{\ell}^2 \int_0^T \|G_{\ell}(s)\|_{\mathbf{H}}^2 ds \right). \end{aligned}$$

Proof. Given $v_0 \in \mathbf{H}$ and $\psi \in L_2((0, T); \mathbf{X}')$, consider a deterministic evolution equation

$$(3.24) \quad v(t) = v_0 + \int_0^t (\mathcal{A}v(s) + \psi(s))ds.$$

By definition, $v \in L_2((0, T); \mathbf{X})$ is a solution of (3.24) if equality (3.24) holds in \mathbf{X}' for every $t \in [0, T]$.

It is known [37, Theorem 3.1.4] that, if the operator \mathcal{A} is strongly parabolic, then

- The operator \mathcal{A} generates a semigroup $\Phi = \Phi_t$, $t \geq 0$, in the space \mathbf{H} .
- The semigroup Φ has the following properties:

$$(3.25) \quad \|\Phi_t v\|_{\mathbf{H}}^2 \leq e^{C_A t} \|v\|_{\mathbf{H}}^2, \quad v \in \mathbf{H};$$

$$(3.26) \quad \left\| \int_0^t \Phi_{t-s} f(s) ds \right\|_{\mathbf{H}}^2 \leq \frac{2}{\delta_A} e^{C_A t} \int_0^t \|f(s)\|_{\mathbf{X}'}^2 ds, \quad f \in L_2((0, T); \mathbf{X}').$$

• The solution of (3.24) is unique and can be written as

$$(3.27) \quad v(t) = \Phi_t v_0 + \int_0^t \Phi_{t-s} \psi(s) ds.$$

We use these results to study the system of equations (3.22). It follows by induction on $|\alpha|$ that if $|\alpha| = k$ with the characteristic set $\{(i_1, \ell_1), \dots, (i_k, \ell_k)\}$, and \mathcal{P}_k is the set of all permutations of $\{1, 2, \dots, k\}$, then the solution of (3.22) is unique and is given by

$$(3.28) \quad \begin{aligned} u_\alpha(t) = & \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_k} \int_0^t \int_0^{s_k} \dots \int_0^{s_2} \Phi_{t-s_k} \mathcal{B}_{\ell_{\sigma(k)}} \dots \Phi_{s_2-s_1} \left(\mathcal{B}_{\ell_{\sigma(1)}} u_{(0)}(s_1) \right. \\ & \left. + G_{\ell_{\sigma(1)}}(s_1) \right) (\mathcal{K}_{\ell_{\sigma(k)}} h_{i_{\sigma(k)}})(s_k) \dots (\mathcal{K}_{\ell_{\sigma(1)}} h_{i_{\sigma(1)}})(s_1) ds_1 \dots ds_k. \end{aligned}$$

We then re-write (3.28) as

$$(3.29) \quad u_\alpha(t) = \int_{[0, T]^k} H(t, \ell^{(k)}; s^{(k)}) \bar{h}_\alpha(s^{(k)}) ds_1 \dots ds_k,$$

where

$$(3.30) \quad \begin{aligned} H(t, \ell^{(k)}; s^{(k)}) = & \frac{1}{\sqrt{k!}} \sum_{\sigma \in \mathcal{P}_k} \Phi_{t-s_{\sigma(n)}} \mathcal{B}_{\ell_n} \dots \Phi_{s_{\sigma(2)}-s_{\sigma(1)}} \left(\mathcal{B}_{\ell_1} u_{(0)}(s_{\sigma(1)}) \right. \\ & \left. + G_{\ell_1}(s_{\sigma(1)}) \right) \chi_{s_{\sigma(2)}}(s_{\sigma(1)}) \dots \chi_t(s_{\sigma(k)}), \end{aligned}$$

and

$$(3.31) \quad \bar{h}_\alpha(s^{(k)}) = \frac{1}{\sqrt{\alpha!} k!} \sum_{\sigma \in \mathcal{P}_k} (\mathcal{K}_{\ell_1} h_{i_1})(s_{\sigma(1)}) \dots (\mathcal{K}_{\ell_k} h_{i_k})(s_{\sigma(k)}).$$

From (3.29) and the definition of the function H , we conclude that

$$(3.32) \quad \begin{aligned} \sum_{\substack{\alpha \in \mathcal{J} \\ |\alpha|=k}} \|u_\alpha(t)\|_{\mathbf{H}}^2 \leq & \sum_{\ell_1, \dots, \ell_k=1}^{\infty} \left(\prod_{j=1}^k \mathfrak{K}_{\ell_j}^2 \right) \int_0^t \int_0^{s_k} \dots \int_0^{s_2} \\ & \left\| \Phi_{t-s_k} \mathcal{B}_{\ell_k} \dots \Phi_{s_2-s_1} \left(\mathcal{B}_{\ell_1} u_{(0)}(s_1) + G_{\ell_1}(s_1) \right) \right\|_{\mathbf{H}}^2 ds_1 \dots ds_k. \end{aligned}$$

By (3.22) and (3.27),

$$(3.33) \quad u_{(0)}(t) = \Phi_t u_0 + \int_0^t \Phi_{t-s} F(s) ds,$$

and then the properties (3.25) and (3.26) of the semigroup Φ imply

$$\|\mathcal{B}_\ell u_{(0)}(t) + G_\ell(t)\|_{\mathbf{H}}^2 \leq C_o \left(C_\ell^2 e^{C_A t} \|u_0\|_{\mathbf{H}}^2 + \frac{2}{\delta_A} C_\ell^2 e^{C_A t} \int_0^t \|F(s)\|_{\mathbf{X}'}^2 ds + \|G_\ell(t)\|_{\mathbf{H}}^2 \right),$$

where $C_o = 3$, as the inequality $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ suggests. On the other hand, $C_o = 1$ if $F = G_\ell = 0$. Applying (3.25) repeatedly, we find

$$\begin{aligned}
 & \|\Phi_{t-s_k} \mathcal{B}_{\ell_k} \cdots \Phi_{s_2-s_1} (\mathcal{B}_{\ell_1} u_{(0)}(s_1) + G_{\ell_1}(s_1))\|_{\mathbf{H}}^2 \\
 (3.34) \quad & \leq C_{\ell_k}^2 e^{C_A(t-s_k)} \|\Phi_{s_k-s_{k-1}} \mathcal{B}_{\ell_{k-1}} \cdots \Phi_{s_2-s_1} (\mathcal{B}_{\ell_1} u_{(0)}(s_1) + G_{\ell_1}(s_1))\|_{\mathbf{H}}^2 \leq \dots \\
 & \leq C_o \left(\prod_{j=2}^k C_{\ell_j}^2 \right) e^{C_A t} \left(C_{\ell_1}^2 \|u_0\|_{\mathbf{H}}^2 + \frac{2}{\delta_A} C_{\ell_1}^2 \int_0^T \|F(s)\|_{\mathbf{X}'}^2 ds + \|G_{\ell_1}(s_1)\|_{\mathbf{H}}^2 \right).
 \end{aligned}$$

Inequality (3.23) now follows from (3.32).

Proposition 3.7 is proved. \square

To complete the proof of Theorem 3.6 it remains to note that uniqueness of the chaos solution is equivalent to the uniqueness of the solution of (3.22) and is guaranteed by the strong parabolicity of the operator \mathcal{A} . Then

$$(3.35) \quad \sum_{\alpha \in \mathcal{J}} \|u_\alpha(t)\|_{\mathbf{H}}^2 = \|u_{(0)}(t)\|_{\mathbf{H}}^2 + \sum_{k \geq 1} \sum_{\substack{\alpha \in \mathcal{J} \\ |\alpha| = k}} \|u_\alpha(t)\|_{\mathbf{H}}^2,$$

and (3.21) follows from (3.23). \square

As a first step toward studying the approximation of the chaos solution, we get

Corollary 3.8. Let $F = G_\ell = 0$ and, for $N \geq 1$, define

$$(3.36) \quad u_N(t) = \sum_{\substack{\alpha \in \mathcal{J} \\ |\alpha| \leq N}} u_\alpha(t) \xi_\alpha.$$

Then

$$(3.37) \quad \sup_{0 < t < T} \mathbb{E} \|u(t) - u_N(t)\|_{\mathbf{H}}^2 \leq \frac{(C_B T)^{N+1}}{(N+1)!} e^{(C_A + C_B)T} \|u_0\|_{\mathbf{H}}^2.$$

Proof. We have

$$(3.38) \quad \mathbb{E} \|u(t) - u_N(t)\|_{\mathbf{H}}^2 = \sum_{\substack{\alpha \in \mathcal{J} \\ |\alpha| \geq N+1}} \|u_\alpha(t)\|_{\mathbf{H}}^2,$$

because

$$(3.39) \quad \mathbb{E}(\xi_\alpha \xi_\beta) = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Then (3.37) follows from (3.23). \square

Remark 3.9. The conclusions of the theorem are valid even if the operators \mathcal{A} , \mathcal{B}_ℓ depend on time in a sufficiently regular way, as long as Assumptions **A1** and **A3** hold uniformly in $t \in [0, T]$.

There are at least two open problems related to the chaos solution of equation (3.1):

- (1) To find out whether the solution belongs to

$$L_2(\Omega; L_2(0, T); X)) \bigcap L_2(\Omega; C((0, T); H));$$

this is true when every \mathfrak{X}_ℓ is a white noise over $L_2((0, T))$ [28, Theorem 3.8].

- (2) To establish existence of the solution when the operators \mathcal{B}_ℓ are unbounded on \mathbf{H} .

4. APPROXIMATION OF THE CHAOS SOLUTION

4.1. One-step Approximation.

4.1.1. *Motivation.* If \mathbf{V} is a Hilbert space, $f \in \mathbf{V}$, and $\{m_k, k \geq 1\}$ is an orthonormal basis in \mathbf{V} , then

$$\sum_{k \geq 1} |(f, m_k)_{\mathbf{V}}|^2 < \infty,$$

but nothing can be said about the rate of this convergence, that is, about the rate at which

$$\sum_{k=n}^{\infty} |(f, m_k)_{\mathbf{V}}|^2$$

tends to zero as $n \rightarrow \infty$: taking $\mathbf{V} = L_2((0, T))$ with a trigonometric basis, one can construct a function for which this convergence will be arbitrarily slow.

In the study of the chaos solution, we are facing a similar problem. In fact, the underlying Hilbert space is the space of square integrable random processes, the study of the rate of convergence for the “natural” approximations of the chaos solution is reduced to the analysis of certain Fourier series in $L_2((0, T))$.

As an illustration, consider the following equation:

$$(4.1) \quad u(t, x) = u_0 + \int_0^t u_{xx}(s, x) ds + \mathfrak{X}(\chi_t h(\cdot, x) u(\cdot, x)), \quad t \geq 0, \quad x \in \mathbb{R},$$

where \mathfrak{X} is a colored noise on $L_2((0, T))$ with representation (\mathcal{K}, W) .

If $u_0 \in L_2(\mathbb{R})$ is non-random, and $h = h(t, x)$ is a bounded non-random function, then Theorem 3.6 on page 8 implies that (4.1) has a unique square-integrable chaos solution $u(t, x) = \sum_{\alpha \in \mathcal{I}} u_\alpha(t, x) \xi_\alpha$, where

$$(4.2) \quad \begin{aligned} \frac{\partial u_{(0)}}{\partial t} &= \frac{\partial^2 u_{(0)}}{\partial x^2}, \quad u_{(0)}(0, x) = u_0(x), \\ \frac{\partial u_\alpha}{\partial t} &= \frac{\partial^2 u_\alpha}{\partial x^2} + \sum_{k=1}^{\infty} \sqrt{\alpha_k} h u_{\alpha - \epsilon_k} \mathcal{K} m_k, \quad u_\alpha(0, x) = 0, \quad |\alpha| > 0, \end{aligned}$$

and $\{m_k, k \geq 1\}$ is an orthonormal basis in $L_2((0, T))$; with only one colored noise driving the equation, every multi-index has the form $\alpha = \{\alpha_1, \alpha_2, \dots\}$.

Denoting the heat semigroup by Φ_t , we find

$$(4.3) \quad u_{(0)}(t) = \Phi_t u_0(x)$$

and

$$(4.4) \quad u_{\epsilon_k}(t, x) = \int_0^t \Phi_{t-s} h \Phi_s u_0(s, x) (\mathcal{K} m_k)(s) ds.$$

Let us define an approximation $u_1^n(t, x)$ of $u(t, x)$ by

$$(4.5) \quad u_1^n(t, x) = u_{(0)}(t, x) + \sum_{k=1}^n u_{\epsilon_k}(t, x) \xi_k.$$

What can we say about the quality of this approximation? For example, can we find a bound on $\sup_{0 < t < T} \mathbb{E} \|u - u_1^n\|_{L_2(\mathbb{R})}^2(t)$ in terms of n and T ?

Since $u(t, x) = \sum_{\alpha \in \mathcal{I}} u_\alpha(t, x) \xi_\alpha$ and (3.39) holds, we have

$$(4.6) \quad \mathbb{E} \|u - u_1^n\|_{L_2(\mathbb{R})}^2(t) = \sum_{\alpha \in \mathcal{I}, |\alpha| > 1} \|u_\alpha\|_{L_2(\mathbb{R})}^2(t) + \sum_{k=n+1}^{\infty} \|u_{\epsilon_k}\|_{L_2(\mathbb{R})}^2(t).$$

Using the properties of the heat semigroup on \mathbb{R} and Corollary 3.8 on page 10, we find

$$(4.7) \quad \sup_{0 < t < T} \sum_{\alpha \in \mathcal{I}, |\alpha| > 1} \|u_\alpha\|_{L_2(\mathbb{R})}^2(t) \leq (C_B T)^2 e^{C_B T},$$

where $C_B = \|\mathcal{K}\| \sup_{t, x} |h(t, x)|^2$.

As a result, to find the quality of the approximation, we need to find the rate of convergence of the series $\sum_{k=1}^{\infty} \|u_{\epsilon_k}\|_{L_2(\mathbb{R})}^2(t)$. This rate of convergence is determined by the rate of decay, as $k \rightarrow \infty$, of $\|u_{\epsilon_k}\|_{L_2(\mathbb{R})}^2(t)$, and, as equality (4.4) suggests, one way to determine this rate is to integrate by parts. Accordingly, setting

$$(4.8) \quad \widetilde{M}_k(t) = \int_0^t (\mathcal{K} m_k)(s) ds,$$

and using the properties of the heat semi-group, we get

$$(4.9) \quad \begin{aligned} u_{\epsilon_k}(t, x) &= \Phi_{t-s} (h(s, \cdot) u_{(0)}(s, \cdot))(x) \widetilde{M}_k(s) \Big|_{s=0}^{s=t} \\ &\quad - \int_0^t \left(\Phi_{t-s} h(\Phi_s u_0)_{xx} - (\Phi_{t-s} h \Phi_s u_0)_{xx} \right) \widetilde{M}_k(s) ds. \end{aligned}$$

Note that (4.8) and the Cauchy-Schwartz inequality imply

$$(4.10) \quad |\widetilde{M}_k(t)| \leq C \sqrt{t},$$

and so $\widetilde{M}_k(0) = 0$. Still, to advance our study of the rate of convergence any further, we need

- (1) additional regularity of u_0 and h ;
- (2) a rather detailed information about the functions \widetilde{M}_k .

If we indeed assume all the necessary regularity of u_0 and h , then (4.9), together with the Cauchy-Schwartz inequality, implies

$$(4.11) \quad \|u_{\epsilon_k}\|_{L_2(\mathbb{R})}^2(t) \leq C_1 \|u_0\|_{L_2(\mathbb{R})}^2 |\widetilde{M}_k(t)|^2 + C_2 \left\| \frac{\partial^2 u_0}{\partial x^2} \right\|_{L_2(\mathbb{R})}^2 t \int_0^t |\widetilde{M}_k(s)|^2 ds.$$

To continue, assume that

$$(4.12) \quad m_1(s) = \frac{1}{\sqrt{T}}; \quad m_k(t) = \sqrt{\frac{2}{T}} \cos\left(\frac{\pi(k-1)t}{T}\right), \quad k > 1; \quad 0 \leq t \leq T.$$

With this choice of the basis, we use (4.8) to find that, for $k > 1$,

(1) If \mathfrak{X} is white noise ($\mathcal{K} = I$), then

$$(4.13) \quad \widetilde{M}_k(t) = \frac{\sqrt{2T}}{\pi(k-1)} \sin\left(\frac{\pi(k-1)t}{T}\right);$$

(2) If \mathfrak{X} is the Ornstein-Uhlenbeck noise with parameter b , then

$$(4.14) \quad \begin{aligned} \widetilde{M}_k(t) = & \frac{\sqrt{2T^3}}{b^2 T^2 + (k-1)^2 \pi^2} \left(b \cos\left(\frac{\pi(k-1)t}{T}\right) - b e^{-bt} \right. \\ & \left. + \frac{(k-1)\pi}{T} \sin\left(\frac{\pi(k-1)t}{T}\right) \right); \end{aligned}$$

(3) If \mathfrak{X} the H -fractional white noise and $1/2 < H < 1$, then

$$(4.15) \quad |\widetilde{M}_k(t)| \leq \frac{C(H) t^{2H-1} T^{1-H}}{k^{\frac{3}{2}-H}}$$

for some number $C(H)$ depending only on H .

Relation (4.11) suggests that the rate of convergence will be quite different for different \mathfrak{X} , and we are essentially forced to make the following assumptions about the functions \widetilde{M}_k :

$$(4.16) \quad \sup_{0 \leq t \leq T} |\widetilde{M}_k(t)|^2 \leq \widetilde{C} \frac{T^\delta}{k^\gamma} \quad \text{for } \delta > 0, \gamma > 1, \widetilde{C} > 0,$$

$$(4.17) \quad |\widetilde{M}_k(T)|^2 \leq \widetilde{C} \frac{T^{\delta_1}}{k^{\gamma_1}} \quad \text{for } \delta_1 > 0, \gamma_1 > 1, \widetilde{C} > 0.$$

In both (4.16) and (4.17), the number \widetilde{C} should not depend on T or k .

It is enough to have (4.16) and (4.17) for *some* orthonormal basis $\{m_k, k \geq 1\}$ in $L_2((0, T))$, but for now the cosine basis (4.12) is the only example when these assumptions can be verified. Below, we summarize the results for the white noise W , fractional white noise W^H , $1/2 < H < 1$, and the Ornstein-Uhlenbeck noise U_b , $b > 0$, when the cosine basis (4.12) is used.

Remark 4.1. (a) In the case of W , with m_k as in (4.12), we have $\widetilde{M}_k(T) = 0$ for all $k \geq 2$, and then indeed any choice of δ_1, γ_1 will work in (4.17).

TABLE 1.

	δ	γ	δ_1	γ_1
W	1	2	Any	Any
W^H	$2H$	$3 - 2H$	$2H$	$3 - 2H$
U_b	1	2	3	4

(b) Inequality (4.11) shows that assumptions (4.16) and (4.17) are close to necessary for the analysis of convergence of the chaos expansion.

(c) As equality (4.9) suggests, no further integration by parts will, in general, improve the rate of convergence.

Under assumption (4.16), we conclude from (4.11) that, for $n > 1$,

$$(4.18) \quad \sup_{0 < t < T} \sum_{k=n}^{\infty} \|u_{\epsilon_k}\|_{L_2(\mathbb{R})}^2(t) \leq \tilde{C}_1 \|u_0\|_{L_2(\mathbb{R})}^2 \frac{T^\delta}{n^{\gamma-1}} + \tilde{C}_2 \left\| \frac{\partial^2 u_0}{\partial x^2} \right\|_{L_2(\mathbb{R})}^2 \frac{T^{\delta+2}}{n^{\gamma-1}}.$$

With assumption (4.17), we also get

$$(4.19) \quad \sum_{k=n}^{\infty} \|u_{\epsilon_k}\|_{L_2(\mathbb{R})}^2(T) \leq \tilde{C}_1 \|u_0\|_{L_2(\mathbb{R})}^2 \frac{T^{\delta_1}}{n^{\gamma_1-1}} + \tilde{C}_2 \left\| \frac{\partial^2 u_0}{\partial x^2} \right\|_{L_2(\mathbb{R})}^2 \frac{T^{\delta+2}}{n^{\gamma-1}}.$$

Inequality (4.18) establishes an approximation error bound uniformly over the time interval $(0, T)$, while (4.19) gives the bound only at the end point. If T is small, and if we can take $\delta_1 > \delta$, $\gamma_1 > \gamma$, which is the case for $\mathfrak{X} = \mathfrak{B}$ and $\mathfrak{X} = U_b$, then (4.19) provides a better error bound than (4.18) and is more suitable for analyzing a step-by-step approximation.

By combining (4.7) with either (4.18) or (4.19), we will get the overall bound on the approximation error; for white, fractional, or Ornstein-Uhlenbeck noise, we also use Table 1. For example, when $\mathfrak{X} = W^H$, $1/2 < H < 1$, we have $\|\mathcal{K}\|^2 \leq C(H)T^{2H-1}$ (see Example 2.5, page 3) and therefore, for $T \leq 1$,

$$(4.20) \quad \sup_{0 < t < T} \mathbb{E} \|u - u_1^n\|_{L_2(\mathbb{R})}^2(t) \leq C^* \left(T^{4H} + \frac{T^{2H}}{n^{2-2H}} \right).$$

In the next section, we extend this result to more general equations and more general approximations.

4.1.2. *Truncation of the S-system.* Consider the following evolution equation:

$$(4.21) \quad u(t) = u_0 + \int_0^t \mathcal{A}u(s)ds + \sum_{\ell=1}^{\infty} \mathfrak{X}_{\ell}(\chi_t \mathcal{B}_{\ell}u),$$

and assume that this equation has a unique square-integrable chaos solution u in a normal triple $(\mathbf{X}, \mathbf{H}, \mathbf{X}')$ of Hilbert spaces. We also assume that u has chaos expansion

$$(4.22) \quad u(t) = \sum_{\alpha \in \mathcal{J}} u_{\alpha}(t) \xi_{\alpha}.$$

As before, we assume that u_0 is deterministic and every colored noise \mathfrak{X}_{ℓ} has a representation $(\mathcal{K}_{\ell}, W_{\ell})$.

The first step is to find a general method of constructing an approximation of u given the expansion (4.22). A natural approximation is

$$(4.23) \quad \bar{u}(t) = \sum_{\alpha \in \bar{\mathcal{J}}} u_{\alpha}(t) \xi_{\alpha},$$

where $\bar{\mathcal{J}}$ is a *finite* subset of \mathcal{J} . To control the size of this finite set, we use three characteristics of a multi-index:

$$|\alpha| = \sum_{k,\ell} \alpha_{k\ell}, \quad \varpi(\alpha) = \max\{k : \alpha_{k\ell} > 0\}, \quad d(\alpha) = \max\{\ell : \alpha_{k\ell} > 0\}.$$

For example, if

$$\alpha = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 3 & 0 & 0 \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

then $|\alpha| = 1 + 1 + 3 + 1 + 2 + 1 = 9$, $\varpi(\alpha) = 8$, $d(\alpha) = 3$. We call $|\alpha|$ the **length of the multi-index**, $\varpi(\alpha)$ the **order of the multi-index**, and $d(\alpha)$ the **dimension of the multi-index**. Then the set

$$\mathcal{J}_N^{n,r} = \{\alpha \in \mathcal{J} : |\alpha| \leq N, \varpi(\alpha) \leq n, d(\alpha) \leq r\}$$

is finite, with no more than $(nr)^N$ elements. Note that the sets $\mathcal{J}_N = \{\alpha \in \mathcal{J} : |\alpha| \leq N\}$ is always infinite, and the set $\mathcal{J}_N^n = \{\alpha \in \mathcal{J} : |\alpha| \leq N, \varpi(\alpha) \leq n\}$ is infinite if and only if there are infinitely many noises in the equation. Accordingly, we define three approximations of u :

$$(4.24) \quad u_N(t) = \sum_{\alpha \in \mathcal{J}_N} u_{\alpha}(t) \xi_{\alpha}, \quad u_N^n(t) = \sum_{\alpha \in \mathcal{J}_N^n} u_{\alpha}(t) \xi_{\alpha}, \quad u_N^{n,r}(t) = \sum_{\alpha \in \mathcal{J}_N^{n,r}} u_{\alpha}(t) \xi_{\alpha}$$

Of the three, only $u_N^{n,r}(t)$ is *computable*, being a sum of finitely many terms. Consequently, our goal is to find a bound on $\sup_{0 < t < T} \mathbb{E} \|u(t) - u_N^{n,r}(t)\|_{\mathbf{H}}$. Recall (see (3.14) on

page 7) that the coefficients u_α satisfy the S-system

$$(4.25) \quad \begin{aligned} u_{(0)}(t) &= u_0 + \int_0^t \mathcal{A}u_{(0)}(s)ds, \quad |\alpha| = 0; \\ u_\alpha(t) &= \int_0^t \mathcal{A}u_\alpha(s)ds + \sum_{k,\ell=1}^{\infty} \sqrt{\alpha_{k\ell}} \int_0^t \mathcal{B}_\ell u_{\alpha-\epsilon_{k\ell}}(s) (\mathcal{K}m_k)(s)ds, \quad |\alpha| > 0. \end{aligned}$$

By orthogonality of ξ_α for different α , we have

$$(4.26) \quad \begin{aligned} \mathbb{E}\|u(t) - u_N^{n,r}(t)\|_{\mathbf{H}}^2 &= \sum_{\alpha \notin \mathcal{J}_N^{n,r}} \mathbb{E}\|u_\alpha(t)\|_{\mathbf{H}}^2 = \sum_{\alpha \in \mathcal{J} \setminus \mathcal{J}_N} \mathbb{E}\|u_\alpha(t)\|_{\mathbf{H}}^2 \\ &+ \sum_{\alpha \in \mathcal{J}_N \setminus \mathcal{J}_N^n} \mathbb{E}\|u_\alpha(t)\|_{\mathbf{H}}^2 + \sum_{\alpha \in \mathcal{J}_N^n \setminus \mathcal{J}_N^{n,r}} \mathbb{E}\|u_\alpha(t)\|_{\mathbf{H}}^2, \end{aligned}$$

where \setminus denotes the difference of two sets. In other words, we have an analogue of the Pithagorean theorem:

$$(4.27) \quad \begin{aligned} \mathbb{E}\|u(t) - u_N^{n,r}(t)\|_{\mathbf{H}}^2 &= \mathbb{E}\|u(t) - u_N(t)\|_{\mathbf{H}}^2 + \mathbb{E}\|u_N(t) - u_N^n(t)\|_{\mathbf{H}}^2 \\ &+ \mathbb{E}\|u_N^n(t) - u_N^{n,r}(t)\|_{\mathbf{H}}^2, \end{aligned}$$

and Corollary 3.8, page 10, provides an estimate for $\mathbb{E}\|u(t) - u_N(t)\|_{\mathbf{H}}^2$.

As we saw in the previous section, to estimate $\mathbb{E}\|u_N(t) - u_N^n(t)\|_{\mathbf{H}}^2$, we need to assume (4.16) and (4.17) together with additional regularity of the initial condition u_0 and the operators \mathcal{A} , \mathcal{B}_ℓ . To formulate this regularity we need some additional constructions.

Let \mathbf{H}^r , $r \in \mathbb{R}$ be a scale of Hilbert spaces, or a **Hilbert scale** [25, Section VI.1.10] with the property that $\mathbf{H}^0 = \mathbf{H}$, $\mathbf{H}^1 = \mathbf{X}$, $\mathbf{H}^{-1} = \mathbf{X}'$. A typical example of such a scale is the collection of the Sobolev spaces

$$(4.28) \quad \mathbf{H}^r(\mathbb{R}^d) = \left\{ f : \int_{\mathbb{R}^d} |\hat{f}(y)|^2 (1 + |y|^2)^r dy < \infty \right\},$$

where \hat{f} is the Fourier transform of f .

To generalize the computations that lead to (4.11), we make the following assumptions:

$$(4.29) \quad \mathbb{E}\|u_0\|_{\mathbf{H}^2}^2 < \infty,$$

$$(4.30) \quad \|\mathcal{A}v\|_{\mathbf{H}}^2 \leq C_{02}\|v\|_{\mathbf{H}^2}^2, \quad \|\Phi_t v\|_{\mathbf{H}^j}^2 \leq e^{C_A t} \|v\|_{\mathbf{H}^j}^2, \quad j = 0, 2,$$

$$(4.31) \quad \mathcal{B}_\ell \mathcal{B}_n = \mathcal{B}_n \mathcal{B}_\ell \text{ for all } \ell, n,$$

$$(4.32) \quad \|\mathcal{B}_\ell v\|_{\mathbf{H}^2}^2 \leq C_{1,\ell}^2 \|v\|_{\mathbf{H}^2}^2, \quad v \in \mathbf{H}^2, \text{ and } \sum_{\ell=1}^{\infty} C_{1,\ell}^2 \mathcal{R}_\ell^2 = C_{1,B} < \infty,$$

where Φ_t is the semi-group generated by \mathcal{A} .

Theorem 4.2. *Assume that*

- **A1–A5** hold (see page 7);
- $F(t) = 0$ and $G_\ell = 0$;

- (4.16), (4.17) hold for all \mathfrak{X}_ℓ so that the numbers $\delta, \delta_1, \gamma, \gamma_1$, do not depend on ℓ ;
- (4.29)–(4.32) hold.

Then

$$(4.33) \quad \sup_{0 < t < T} \mathbb{E} \|u_N(t) - u_N^n(t)\|_{\mathbf{H}}^2 \leq C_{1,B} e^{(C_A + \overline{C}_B)T} \left(C_B \frac{T^\delta}{n^{\gamma-1}} \mathbb{E} \|u_0\|_{\mathbf{H}}^2 + C_{02} C_{1,B} \frac{T^{\delta+2}}{n^{\gamma-1}} \mathbb{E} \|u_0\|_{\mathbf{H}^2}^2 \right)$$

and

$$(4.34) \quad \mathbb{E} \|u_N(T) - u_N^n(T)\|_{\mathbf{H}}^2 \leq C_{1,B} e^{(C_A + \overline{C}_B)T} \left(C_B \frac{T^{\delta_1}}{n^{\gamma_1-1}} \mathbb{E} \|u_0\|_{\mathbf{H}}^2 + C_{02} C_{1,B} \frac{T^{\delta+2}}{n^{\gamma-1}} \mathbb{E} \|u_0\|_{\mathbf{H}^2}^2 \right),$$

where $\overline{C}_B = \max(C_B, C_{1,B})$.

Proof. The argument is based on integration by parts in the representation of $u_\alpha(t)$ (see (3.29) on page 9); while the idea and the end result are essentially identical to (4.9), the computations are rather long. An interested reader can recover these computations following [27], where each \mathfrak{X} is white noise. \square

Remark 4.3. Similar to [27], a bound on $\mathbb{E} \|u_N(t) - u_N^n(t)\|_{\mathbf{H}}^2$ can be derived without condition (4.31), that is, if the operators \mathcal{B}_ℓ do not commute.

Finally, we derive a bound on $\mathbb{E} \|u_N^n(t) - u_N^{n,r}(t)\|_{\mathbf{H}}^2$.

Theorem 4.4. *Assume that*

- **A1–A5** hold;
- $F(t) = 0$ and $G_\ell = 0$;

Define the sequence $\varepsilon = \varepsilon(r)$, $r = 1, 2, \dots$ by

$$(4.35) \quad \sum_{\ell=r+1}^{\infty} \mathfrak{K}_\ell^2 C_\ell^2 = \varepsilon(r).$$

Then

$$(4.36) \quad \sup_{0 < t < T} \mathbb{E} \|u_N^n(t) - u_N^{n,r}(t)\|_{\mathbf{H}}^2 \leq \varepsilon(r) T e^{(C_A + C_B)T} \mathbb{E} \|u_0\|_{\mathbf{H}}^2.$$

Proof. We have by (3.32) on page 9

$$\begin{aligned} \mathbb{E} \|u_N^n(t) - u_N^{n,r}(t)\|_{\mathbf{H}}^2 &\leq \sum_{k=1}^N \sum_{j=1}^k \sum_{\ell_j=r+1}^{\infty} \sum_{\ell_1, \dots, \ell_k \geq 1}^{(j)} \int_0^t \int_0^{s_k} \dots \int_0^{s_2} \\ &\quad \|\Phi_{t-s_k} \mathcal{B}_{\ell_k} \dots \Phi_{s_2-s_1} \mathcal{B}_{\ell_1} u_{(0)}(s_1)\|_{\mathbf{H}}^2 ds_1 \dots ds_k, \end{aligned}$$

where the summation $\sum_{\ell_1, \dots, \ell_k \geq 1}^{(j)}$ omits the index ℓ_j . Using assumptions the theorem, we conclude that

$$\mathbb{E}\|u_N^n(t) - u_N^{n,r}(t)\|_{\mathbf{H}}^2 \leq e^{C_A t} \left(\sum_{\ell=r+1}^{\infty} \mathfrak{K}_{\ell}^2 C_{\ell}^2 \right) \left(\sum_{k=1}^{\infty} k \frac{C_B^{k-1} t^k}{k!} \right) \mathbb{E}\|u_0\|_{\mathbf{H}}^2,$$

which implies (4.36). \square

Remark 4.5. If there are finitely many \mathfrak{X}_{ℓ} , and all of them are included in the approximation, then $\varepsilon(r) = 0$ and $u_N^n(t) = u_N^{n,r}(t)$.

Combining the results of Corollary 3.8 and Theorems 4.2 and 4.4, we get the overall error bound:

$$(4.37) \quad \sup_{0 < t < T} \mathbb{E}\|u(t) - u_N^{n,r}(t)\|_{\mathbf{H}}^2 \leq C(T) \left(\frac{(TC_B)^{N+1}}{(N+1)!} \mathbb{E}\|u_0\|_{\mathbf{H}}^2 \right. \\ \left. + \frac{T^{\delta}}{n^{\gamma-1}} \mathbb{E}\|u_0\|_{\mathbf{H}}^2 + \frac{T^{\delta+2}}{n^{\gamma-1}} \mathbb{E}\|u_0\|_{\mathbf{H}^2}^2 + T\varepsilon(r) \mathbb{E}\|u_0\|_{\mathbf{H}}^2 \right),$$

where $\lim_{T \rightarrow 0} C(T) > 0$. A similar bound holds for $\mathbb{E}\|u(T) - u_N^{n,r}(T)\|_{\mathbf{H}}^2$:

$$(4.38) \quad \mathbb{E}\|u(T) - u_N^{n,r}(T)\|_{\mathbf{H}}^2 \leq C(T) \left(\frac{(TC_B)^{N+1}}{(N+1)!} \mathbb{E}\|u_0\|_{\mathbf{H}}^2 \right. \\ \left. + \frac{T^{\delta_1}}{n^{\gamma_1-1}} \mathbb{E}\|u_0\|_{\mathbf{H}}^2 + \frac{T^{\delta+2}}{n^{\gamma-1}} \mathbb{E}\|u_0\|_{\mathbf{H}^2}^2 + T\varepsilon(r) \mathbb{E}\|u_0\|_{\mathbf{H}}^2 \right),$$

Example 4.6. Consider the equation

$$u(t) = u_0 + \int_0^t u_{xx}(s) ds + \mathfrak{X}(\chi_t u), \quad 0 \leq t \leq T, \quad x \in \mathbb{R}.$$

With only one noise driving the equation, we have $u_N^{n,r} = u_N^n$. Also, $\mathbf{H}^r = \mathbf{H}^r(\mathbb{R})$ is the Sobolev space (4.28) and $\mathbf{H} = L_2(\mathbb{R})$.

(a) If \mathfrak{X} is an Ornstein-Uhlenbeck noise with parameter b , then $C_B = (1 + \sqrt{bT})^2$, $\delta = 1$, $\gamma = 2$ (see Table 1). Inequality (4.37) becomes

$$(4.39) \quad \sup_{0 < t < T} \mathbb{E}\|u(t) - u_N^n(t)\|_{\mathbf{H}}^2 \leq C_b(T) \left(\frac{(1 + \sqrt{bT})^{2N+2} T^N}{(N+1)!} \mathbb{E}\|u_0\|_{\mathbf{H}}^2 \right. \\ \left. + \frac{T}{n} \mathbb{E}\|u_0\|_{\mathbf{H}}^2 + \frac{T^3}{n} \mathbb{E}\|u_0\|_{\mathbf{H}^2}^2 \right),$$

where $\limsup_{T \rightarrow 0} C_b(T) > 0$.

(b) If \mathfrak{X} is an H -fractional white noise with $H \in (1/2, 1)$, then $C_B = C_1(H)T^{2H-1}$, where

$$(4.40) \quad C(H) = \frac{H(2H-1) \Gamma(H - \frac{1}{2})}{\Gamma(H + \frac{1}{2})};$$

see (2.3) on page 4. Also, $\delta = 2H$, $\gamma = 3 - 2H$ (see Table 1). Inequality (4.37) becomes

$$(4.41) \quad \sup_{0 < t < T} \mathbb{E} \|u(t) - u_N^{n,r}(t)\|_{\mathbf{H}}^2 \leq C_H(T) \left(\frac{(C_1(H))^{N+1} T^{2H(N+1)}}{(N+1)!} \mathbb{E} \|u_0\|_{\mathbf{H}}^2 \right. \\ \left. + \frac{T^{2H}}{n^{2-2H}} \mathbb{E} \|u_0\|_{\mathbf{H}}^2 + \frac{T^{2H+2}}{n^{2-2H}} \mathbb{E} \|u_0\|_{\mathbf{H}^2}^2 \right),$$

where $\limsup_{T \rightarrow 0} C_H(T) > 0$.

4.2. Step-by-Step Approximation.

4.2.1. *Motivation.* In the previous section, we constructed an approximate solution for equation (4.21) on the time interval $[0, T]$ and derived an error bound. The error bound suggests that the quality of the approximation improves for small values of T . To construct the approximation for large values of T , it is natural to use a step-by-step method.

The main idea of the step-by-step method is as follows. Let Ψ_t be the solution operator for a homogenous linear evolution equation, that is, given an initial condition u_0 , $u(t) = \Psi_t u_0$ is the solution of the equation at time t . If the equation is time-homogeneous (has no explicit dependence on time, such as (4.21)) and the solution is unique, then the solution operator has the semi-group property:

$$(4.42) \quad \Psi_t u_0 = \Psi_{t-s} \Psi_s u_0, \quad t > s > 0.$$

If, for each $t > 0$, $u(t)$ is an element of an infinite-dimensional Hilbert space X with norm $\|\cdot\|$, then a one-step approximate solution $\bar{u}(t)$ can be constructed by

$$\bar{u}(t) = \Pi^N \Psi_t u_0,$$

where Π^N is an orthogonal projection on an N -dimensional sub-space of X . Assume that the approximation is of order $p/2$ in time for some $p > 1$:

$$(4.43) \quad \|\bar{u}(t) - u(t)\|^2 = \|(I - \Pi^N) \Psi_t u_0\|^2 \leq C t^p \|u_0\|^2,$$

where I denotes the identity operator. To construct a multi-step approximation on $[0, T]$, let $0 = t_0 < t_1 < \dots < t_K = T$ be a uniform partition of $[0, T]$ with step τ . Then define \bar{u}_i , $i = 0, \dots, K$, recursively as follows:

$$(4.44) \quad \bar{u}_0 = u_0, \quad \bar{u}_{i+1} = \Pi^N \Psi_\tau \bar{u}_i.$$

For simplicity, we assume that the initial condition is not approximated, and concentrate only on the effects of approximating the solution operator Ψ_t . Then, by linearity,

$$(4.45) \quad u(t_i) - \bar{u}_i = \Psi_\tau u(t_{i-1}) - \Pi^N \Psi_\tau \bar{u}_{i-1} = (I - \Pi^N) \Psi_\tau u_{t_{i-1}} + \Pi^N \Psi_\tau (u_{t_{i-1}} - \bar{u}_{i-1}).$$

By orthogonality, we find

$$(4.46) \quad \|u(t_i) - \bar{u}_i\|^2 = \|\Psi_\tau u(t_{i-1}) - \Pi^N \Psi_\tau \bar{u}_{i-1}\|^2 \\ = \|(I - \Pi^N) \Psi_\tau u_{t_{i-1}}\|^2 + \|\Pi^N \Psi_\tau (u_{t_{i-1}} - \bar{u}_{i-1})\|^2.$$

Let $\Delta_i = \|u(t_i) - \bar{u}_i\|^2$. Then (4.46) and (4.43) imply

$$(4.47) \quad \Delta_i \leq C\tau^p \|u_{t_{i-1}}\|^2 + \|\Psi_\tau(u_{\tau_{i-1}} - \bar{u}_{i-1})\|^2.$$

In many situations, the semi-group Ψ_t satisfies

$$(4.48) \quad \|\Psi_t f\| \leq e^{at} \|f\|$$

for some $a > 0$. In this case, $\|u_{t_{i-1}}\|^2 \leq e^{2aT} \|u_0\|^2$ and (4.47) implies

$$(4.49) \quad \Delta_i \leq C_1 \tau^p \|u_0\|^2 + e^{2a\tau} \Delta_{i-1},$$

or, after applying this inequality repeatedly,

$$(4.50) \quad \Delta_i \leq C_1 \tau^p \|u_0\|^2 \sum_{j=0}^i e^{2a\tau j}.$$

Since

$$(4.51) \quad \sum_{j=0}^i e^{2a\tau j} \leq \sum_{j=0}^K e^{2a\tau j} = \frac{e^{2a(K+1)\tau} - 1}{e^{2a\tau} - 1} \leq \frac{e^{4aT} - 1}{e^{2a\tau} - 1},$$

and $e^{2a\tau} - 1 \geq 2a\tau$, we conclude that

$$(4.52) \quad \Delta_i \leq C_2 \tau^{p-1} \|u_0\|^2,$$

that is,

$$(4.53) \quad \max_{0 \leq i \leq K} \|u(t_i) - \bar{u}_i\|^2 \leq C_2 \tau^{p-1} \|u_0\|^2,$$

where C_2 depends only on T and the semi-group Ψ_t . In other words, the step-by-step approximation has order $(p-1)/2$ in time. The derivation of this result essentially relies on the following:

- (1) an approximation based on an orthogonal projection;
- (2) the property (4.48) of the solution operator.

4.2.2. The Chaos Solution. Let us consider equation (4.21) on page 15. The approximation $u_N^{n,r}(t)$ of the solution is based on an orthogonal projection in the space of square-integrable processes and, by Theorem 3.6, page 8, the solution operator for the equation satisfies (4.48). We can therefore use (4.53) to derive an error bound for the step-by-step approximation of the solution of (4.21).

Let $0 = t_0 < t_1 < \dots < t_K = T$ be a uniform partition of the interval $[0, T]$ with step τ : $t_j = j\tau$, $j = 0, \dots, K$. Let $\{m_k, k \geq 1\}$ be an orthonormal basis in $L_2((0, T))$ and $m_k^j(t) = m_k(t - t_j)(\chi_{t_{j+1}}(t) - \chi_{t_j}(t))$. We define

$$(4.54) \quad \xi_{k\ell}^j = \int_{t_{j-1}}^{t_j} m_k^j(t) dW_\ell(t),$$

and then, for $\alpha \in \mathcal{J}$,

$$(4.55) \quad \xi_\alpha^j = \prod_{k,\ell} \frac{H_{\alpha_{k\ell}}(\xi_{k\ell}^j)}{\sqrt{\alpha_{k\ell}!}}.$$

Note that the random variables $\xi_{k\ell}^i$ and ξ_{pq}^j are independent for different i, j .

If $u = u(t; u_0)$ is the square-integrable chaos solution of the homogeneous equation (4.21) with initial condition u_0 , then, by uniqueness and time homogeneity, we have

$$(4.56) \quad u(t_j; u_0) = u(\tau; u(t_{j-1}, u_0)),$$

which is a particular case of the general relation (4.42). Also, by Theorem 3.6 on page 8,

$$(4.57) \quad \mathbb{E}\|u(t; u_0)\|_{\mathbf{H}}^2 \leq e^{(C_A+C_B)t} \mathbb{E}\|u_0\|_{\mathbf{H}}^2,$$

which is a particular case of (4.48) on page 20.

Next, consider the following modification of the S-system (4.25) from page 16:

$$(4.58) \quad \begin{aligned} u_{(0)}^j(t; f) &= f + \int_{t_{j-1}}^t \mathcal{A}u_{(0)}(s; f)ds, \quad |\alpha| = 0; \\ u_{\alpha}^j(t; f) &= \int_{t_{j-1}}^t \mathcal{A}u_{\alpha}^j(s; f)ds + \sum_{k, \ell=1}^{\infty} \sqrt{\alpha_{k\ell}} \int_{t_{j-1}}^t \mathcal{B}_{\ell} u_{\alpha-\epsilon_{k\ell}}^j(s; f) (\mathcal{K}m_k^j)(s)ds \end{aligned}$$

for $|\alpha| > 0$, where $t_{j-1} \leq t \leq t_j$, $\mathbb{E}\|f\|_{\mathbf{H}}^2 < \infty$, and f is random but independent of ξ_{α}^j , $\alpha \in \mathcal{J}$.

Theorem 4.7. *Assume A1–A5. Then, for $j = 1, \dots, K$,*

$$(4.59) \quad u(t_j; u_0) = \sum_{\alpha \in \mathcal{J}} u_{\alpha}^j(t_j; u(t_{j-1}; u_0)) \xi_{\alpha}^j$$

and

$$(4.60) \quad \mathbb{E}\|u(t_j; u_0)\|_{\mathbf{H}}^2 \leq e^{(C_A+C_B)\tau} \mathbb{E}\|u(t_{j-1}; u_0)\|_{\mathbf{H}}^2.$$

Proof. Since $u(t_{j-1}; u_0)$ is independent of $\{\xi_{\alpha}^j, \alpha \in \mathcal{J}\}$, Theorem 3.6 can be applied on each interval $[t_{j-1}, t_j]$, $j = 1, \dots, K$, with u_0 replaced by $u(t_{j-1}; u_0)$. Then (4.56) becomes (4.59) and (3.21) becomes (4.60). \square

Define the multi-step approximation of u as follows:

$$(4.61) \quad u_N^{n,r}(t_j) = \sum_{\alpha \in \mathcal{J}_N^{n,r}} u_{\alpha}^j(t_j; u_N^{n,r}(t_{j-1})) \xi_{\alpha}^j, \quad j = 1, \dots, K,$$

with $u_N^{n,r}(t_0) = u_0$. The following theorem provides an error bound for this approximation.

Theorem 4.8. *Under the assumptions of Theorems 4.2 and 4.4*

$$(4.62) \quad \begin{aligned} \max_{j=1, \dots, K} \mathbb{E}\|u(t_j; u_0) - u_N^{n,r}(t_j)\|_{\mathbf{H}}^2 &\leq C(T) \left(\frac{(\tau C_B)^N}{(N+1)!} \mathbb{E}\|u_0\|_{\mathbf{H}}^2 \right. \\ &\quad \left. + \frac{\tau^{\delta_1-1}}{n^{\gamma_1-1}} \mathbb{E}\|u_0\|_{\mathbf{H}}^2 + \frac{\tau^{\delta} + 1}{n^{\gamma-1}} \mathbb{E}\|u_0\|_{\mathbf{H}^2}^2 + \varepsilon(r) \mathbb{E}\|u_0\|_{\mathbf{H}}^2 \right). \end{aligned}$$

Proof. Since $u_N^{n,r}(\tau)$ is an orthogonal projection of $u(\tau; u_0)$ on the span of ξ_{α} , $\alpha \in \mathcal{J}_N^{n,r}$, and (4.57) holds, the result follows from (4.38) and (4.53). \square

Example 4.9. Consider the equation

$$u(t) = u_0 + \int_0^t u_{xx}(s)ds + \mathfrak{X}(\chi_t u), \quad 0 \leq t \leq T, \quad x \in \mathbb{R}.$$

With only one noise driving the equation, we have $u_N^{n,r} = u_N^n$. Also, $\mathbf{H}^r = \mathbf{H}^r(\mathbb{R})$ is the Sobolev space (4.28) and $\mathbf{H} = L_2(\mathbb{R})$.

(a) If \mathfrak{X} is an Ornstein-Uhlenbeck noise with parameter b , then $C_B = (1 + \sqrt{b\tau})^2$, $\delta = 1$, $\gamma = 2$, $\delta_1 = 3$, $\gamma_1 = 4$ (see Table 1), so that (4.62) becomes

$$(4.63) \quad \max_{j=1,\dots,K} \mathbb{E} \|u(t_j; u_0) - u_N^{n,r}(t_j)\|_{\mathbf{H}}^2 \leq C_b(T) \left(\frac{(1 + \sqrt{b\tau})^{2N+2} \tau^N}{(N+1)!} \mathbb{E} \|u_0\|_{\mathbf{H}}^2 + \frac{\tau^2}{n^3} \mathbb{E} \|u_0\|_{\mathbf{H}}^2 + \frac{\tau^2}{n} \mathbb{E} \|u_0\|_{\mathbf{H}^2}^2 \right).$$

(b) If \mathfrak{X} is an H -fractional white noises with parameter $H \in (1/2, 1)$, then $C_B = C_1(H)\tau^{2H-1}$ (see (4.40)), $\delta = \delta_1 = 2H$, $\gamma = \gamma_1 = 3 - 2H$ (see Table 1), so that (4.62) becomes

$$(4.64) \quad \max_{j=1,\dots,K} \mathbb{E} \|u(t_j; u_0) - u_N^{n,r}(t_j)\|_{\mathbf{H}}^2 \leq C_H(T) \left(\frac{(C_1(H))^{N+1} \tau^{2H(N+1)-1}}{(N+1)!} \mathbb{E} \|u_0\|_{\mathbf{H}}^2 + \frac{\tau^{2H-1}}{n^{2-2H}} \mathbb{E} \|u_0\|_{\mathbf{H}}^2 + \frac{\tau^{2H+1}}{n^{2-2H}} \mathbb{E} \|u_0\|_{\mathbf{H}^2}^2 \right).$$

ACKNOWLEDGMENT

SVL is grateful to Professor Wendell Fleming for very fruitful discussions, and to the Division of Applied Mathematics at Brown University for hospitality and stimulating atmosphere.

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